The Application of Bolyai's 3+1 Teaching-Learning Tetrahedron Model in Everyday Life
A Bolyai 3+1 tanítási-tanulási tetraédermodell alkalmazása a mindennapokban

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***Abstract*—In our paper we present János Bolyai's teaching model, which is built on the unity of 'know-feel-want', while the free spirit falls towards the body just like space bends around matter. After outlining Bolyai's model, we point out how the learning theories underlying the three major educational methodologies of the last century (behaviourism, cognitivism and constructivism) are combined by the network theory, connectivism, applying Bolyai's methodology. Then we rethink Bolyai's model to show what Bolyai wanted to teach to mankind; the past is within us, and if we do not learn how to use it, it will bring an end to humanity. The problem is in the capacity of human memory. Without training the work memory, the teachers and the students cannot reproduce the 'know-feel-want + body' (3 + 1) model. Finally, we present the proving experiment we conducted with the revised model.**

***Absztrakt*—Előadásunkban bemutatjuk Bolyai János tanítási modelljét, amely a tud-érez-akar egysége, miközben a szabad szellem úgy zuhan a test felé, ahogy az anyag mellett a tér meghajlik. A Bolyai-modell felvázolása után rámutatunk arra, hogy az elmúlt évszázad három jelentős oktatási módszertanának hátterében álló tanuláselméletet (a behaviorizmust, a kognitivizmust, a konstruktivizmust) Bolyai módszertanát alkalmazva hogyan fogja össze a hálózatelmélet, a konnektivizmus, és egy kiinduló modellt nyújtunk a tanításra Bolyai-modellezéssel. Ezután újragondoljuk a Bolyai-modellt, hogy megmutassuk azt, amire Bolyai János meg akarta tanítani az emberiséget: a múlt tapasztalatai itt vannak bennünk, és ha nem tanuljuk meg felhasználni, az az emberiség vége. A probléma az emberi memória képességében van. A munkamemória edzése nélkül a hallgatók, tanárok nem képesek a tud-érez-akar+test (3+1) modellen modellezni. Végül ismertetjük az újragondolt modellel végzett bizonyítási kísérletünket.**

Keywords: Bolyai-model, know-feel-want model, learning theories, application of modelling techniques in education.

Kulcsszavak: Bolyai-modell, tud-érez-akar modell, tanuláselméletek, modellezési módszer alkalmazása az oktatásban.

I. INTRODUCTION

The two authors' age, and, consequently, their teaching experience and system analyst's and modelling practice and knowledge, the information they have about János Bolyai's work differ to a large extent. Still, they both think that they should familiarise their environment and as many people as possible with Bolyai's view of the world and his modelling methodology.

The application of Bolyai's[1] tetrahedron model in individuals', communities' and societies' everyday life would be highly beneficial, because we could reach the necessary uniform knowledge efficiently, by using the same language. Also, we would learn that it is not the truth of one or another individual, community, etc. that should prevail if they cannot make the other party to accept their own truth but we would find the third truth that can be accepted by both parties.

This paper will firstly present Bolyai's system approach in a generally comprehensible way. Then, based on and using Bolyai's modelling, it will point out a few features of today's education. Finally, it will present a new, adapted Bolyai model and a practical experiment to test its use.

II. JÁNOS BOLYAI'S SYSTEM APPROACH AND TEACHING MODEL

A. János Bolyai's system approach

Before discussing his teaching model, let us have an overview of Bolyai's view of the world and his system approach, as it is necessary for us to understand his model.

In his manuscript sheet [2:JB79-1] János Bolyai defined what he meant by his own creation, the Hungarian word 'ür-tan' (meaning the science of the space of Bolyai's geometry, the real knowledge of mankind). *"The science of Ür gives us an answer to the question how one can choose a method from geometry to solve a problem, once they have acquired the knowledge necessary to be able to choose the model and have learnt how to model it."* [1:203-204] He recognised that within the interaction between contradictory structures that are not contradictory

[1] János Bolyai, born in Kolozsvár, 15 December 1802, died in Marosvásárhely, 27 January 1860.

themselves, based on free choice, a third structure exists, which is free from contradiction. *"At this point all you would have to do is try to convince the other party until they are absolutely certain, until both parties have the same belief, or at least accept a third true opinion. Thus, it is interesting, nice and useful to create an independent theory, which is true in itself, in a third true theory!"* [2:JB 72/1]

At the same time, his method is also dialectic, but, unlike the dialectics of Hegel or Engels, János Bolyai does not consider the fight between contradictory relations inevitable. He developed learning patterns for the knowledge revolution of mankind, so that the walls separating people, communities, nations and cultures would crumble.

At first (before 1842), Bolyai sought for the opportunity to examine freedom from contradiction in the development of mathematics. In the 19th century his approach to geometry was (and we could say that even today it is) specific and original. He recognised that geometric space was suitable for transmitting communication between people. A system of axioms constitutes the basis of geometry, on which definitions and postulates are built. Information from the geometric space is transmitted by the solutions of functions and by human languages.

One of his main difficulties was that all human languages contain redundancies; therefore geometric forms cannot be transformed into exact information during communication. Also, communicative relationships are interspersed with vested interests and the individuals and communities in a society are linked by deceptive, untrue information. Since then a significant progress has been achieved in the process of making algorithms. Modelling languages have been created for software development, as well as a large number of programming languages for making software, e.g. protocols for network traffic. There are a lot of languages in which we can conduct unambiguous communication. Having language translating, search and artificial intelligence programmes that make use of large computational, network brandwidth and storage capacity, now it seems feasible for two modellers living in any two places on the Earth to communicate with each other, irrespective of what language they speak.

One of the consequences of the spread of IT is that the teaching and research of geometrics is having a renaissance. Geogebra[2] is an example for that.

According to János Bolyai the inefficiency of communication is just one difficulty. The other, greater one is related to the use of the geometric approach in modelling. For the majority of those constituting society, in grassroots systems, his work, Appendix [3] is almost unknown, therefore the modellers are forced to choose just one geometric model, that of Euclid.

Because of languages being insufficient for communication, in a later period of his life (from 1842) Bolyai started to develop the grammar of the Hungarian language. He thought that the etymons, root words of this language would provide the opportunity to develop a linguistic code system people could master in a short time as a world language and could use it to create their models of reality.

By combining the interpretation of the Hungarian words 'rend' (order) and 'szer' (element, e.g. material, process, information), he created his system approach, in which he interpreted space, matter in space, intellect and the divine world as a system.

"In the Hungarian interpretation of the concept of system the models of two separate wholes appear in the world 'rendszer', where 'szer', which used to have an independent meaning, stands for the real form, independent of the system (the circumscribed entirety of imaginary solid and liquid matter). 'Rend' appears as the set of rules that arrange the system into a whole. Thus, the concept of system can be interpreted as a model in which the separate existence of mind and matter is not applicable." [1:29] Bolyai's attempt to develop a new system of rules of the Hungarian language did not succeed.

His view of the world was also formulated by contemporary scientific theories and the modelling of the practical knowledge of his surroundings.

Scientists in Bolyai's time did not have a system approach, considering that Bertalanffy only described the place and role of systems in sciences in the middle of the 20th century [4]. They did not appreciate János Bolyai's geometry appropriately, they did not accept his modelling method. From his writings stemming from his experiments with his mind – albeit imperfectly – emerges the methodology of the intellect of a researcher of genius.

Thus, János Bolyai traced the common theory of sciences back not to physics but to geometry. In his approach to geometry the physical and the living form of matter gained identical interpretation. He expressed the elements of both by the concept of 'szer' (hereinafter: 'element'). What is more, he also defined that they have an order that co-ordinates material and living 'elements'. This organising principle is the spiritual feature called *know-feel-want*, which merges into one in the attraction of the body. The ability of *know-feel-want* is a feature linked to the human body, which makes it possible to describe the status of the modelled system. The aim of modelling is to create the unity of these three abilities. This is what we call today the mental ability of humans. Today's psychology does not know the modelling method of Bolyai, therefore it does not treat man's ability to *know-feel-want* in a uniform system, either.

Order, however, is not the same in physical and living systems. While systems among physical 'elements' are only created by feeling, the gravitational force, from living 'elements' it is the 'elements' of feeling and will from which animals and plants create systems. When interpreting the 'element' of man, he uses the phrase 'clever animal'. From this we may conclude that man, apart from having the 'elements' of feeling and will, is more than plants and animals, because, using his ability to know something, he can create a system. Thus, the system of man is created in the attraction of *know-feel-want* and the body. Why was it necessary that physical and living matter appear in his view of the world as a concept of system? Because his geometric system carries contradiction among the structures of subsystems that are free from contradiction, and this contradiction is recognisable and applicable for characterising both

[2] GeoGebra: Software for the teaching of mathematics. Its content is related to geometry, algebra and calculus. The application is mainly defined by two aspects: a shape is present both in the form of a formula and as a geometric form. [5]

physical and living systems and for modelling both of them. This view of the world is embraced by few people, even in the 21st century.

As Bolyai put it, (Appendix 15.§ [3]) he built his geometric system from Euclid's geometric system, from the non-Euclidean superstructure and from the absolute (third way) – subsystems, according to today's concepts – into a whole (see Figure 1 below), noting that the non-Euclidean superstructure could be of a thousand kinds.

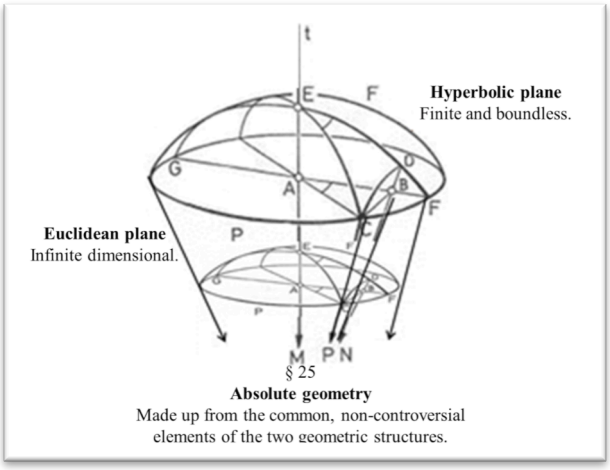


Figure 1: The Euclidean plane, the hyperbolic plane and the absolute geometric plane constructed from them.

As the contradiction-free state of the Euclidean geometry did not reach a final, absolute state, despite the philosophical establishment also confirmed by Hegel, János Bolyai developed another geometric system, the (hyperbolic) non-Euclidean S-superstructure. In the Euclidean space, parallels positioned at the same distance constitute a lattice. Its fundamental unit is the square and the cube. The development of sciences was defined by this lattice in ancient times. In non-Euclidean space parallels are constituted by straights not intersecting each other and pointing at a specific point. In this space, sciences move towards each other as they develop and fill the space between sciences. The fundamental unit is the triangle and the tetrahedron. (In the second half of the 19th century elliptic geometry was also born with the circle and the cylinder as basic shapes but in this paper we do not discuss this.)

So that the contradictory relationship between Euclidean and non-Euclidean geometric structures would not hinder the development of sciences and the empirical knowledge of mankind, Bolyai outlined a third one, an absolute geometric structure. The three structures together are suitable for fulfilling the role of 'yeast' in the development of the scientific and empirical knowledge of mankind.

B. The tetrahedron model of Bolyai's teaching

János Bolyai developed a unified system for the teaching of mankind. He asked questions about the interaction between its elements and answered them. According to certain researchers he recorded his ideas about modelling on the model of teaching and learning on 14000 pages, among which Imre Seebauer came across mainly with pages not considered by other scientists as important and misunderstood. These are the ones where he summarises his modelling methodology.

Thus, in his model, János Bolyai uses the simplest geometric shape, the triangle. This is the geometric shape in his system approach which arranges the regularities recognised by the modeller into a model. As all the geometric forms can be broken down into triangles, Bolyai chose it to be the tool for his modelling. He built another structure on the triangle and he named the tetrahedron as the fundamental unit of his system. The tetrahedron is not a simple geometric shape any longer, but the joint model of physical and living systems. This model is suitable not only for modelling geometry but *"all the sciences will be adequately covered, what is needed in the befitting subject; all the rest will depend on the authority of application (which yields itself rather well understood, permeated by cleverness, nice taste) (per se)"*.[3] [2:JB79/1] (The word *authority* should be interpreted as *teacher*.)

The drawing in the next Figure was very important for him, which is evident from the fact that he also intended it to be on the cover of his work titled *Tan* (Science) [6] as a picture depicting the Universe. The drawing is so general that it was the basis of all of his models, no matter what he was modelling. That is what we do, too, when we intend to evaluate today's teaching-learning structures, applying Bolyai's methodology.

The science of 'ür' (that is the science of the space) = (tetrodom) sheet 4-science can be represented by a tetrahedron that organises the physical and living 'elements' into a system, according to the *know-feel-want* orders in the modelling process. This, in turn, means that by applying the model, by modelling on it, mankind can get to know man, mankind and the Universe. New, true knowledge gained from modelling makes it possible for mankind to organise itself so it can stay alive and happy on Earth and in the Galaxy.

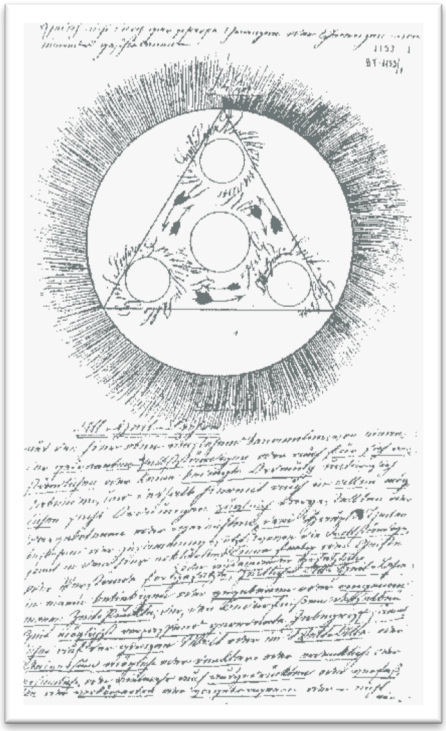


Figure 2: The Figure designed for János Bolyai's work titled *Tan* (Science), 1834. [6]

[3] We did not find anyone who could translate the text from German.

Geometric modelling is realised on the triangle base of the tetrahedron. We map the pattern of reality being examined on this surface. János Bolyai called this base dead, because, if we gain a mathematical picture from geometry, we may perceive that picture as true, that is a result of modelling that is true and cannot be developed further, the ultimate reality.

In the top vertex of the triangle base of Bolyai's tetrahedron we have 'know', the ability to know, which we call explicit knowledge today (see Figure below). By it we mean the knowledge of facts, which we acquire through information. This is what people share within formal frameworks (in schools, trainings, books, etc.). It can be systematised, recorded and easily formulated for others.

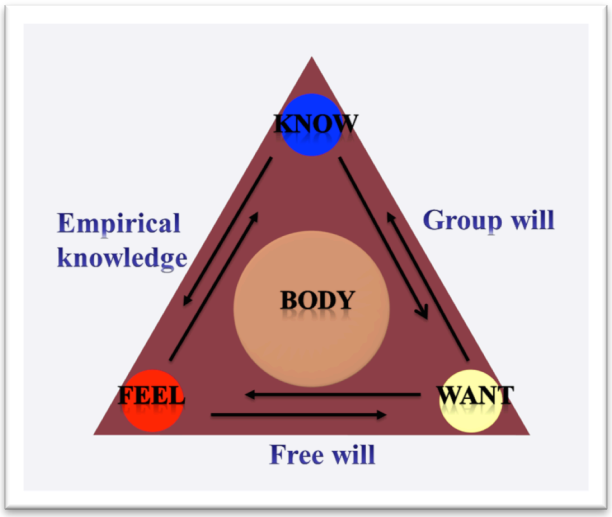
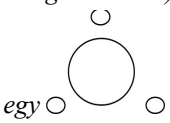


Figure 3: The triangle base of Bolyai's tetrahedron with empirical knowledge, free will and group will indicated.

In the bottom left vertex of the triangle above we have the ability to feel, which today we usually call empirical, implicit knowledge, or, based on Mihály Pollányi's book titled *Személyes tudás (Personal knowledge)* [7], tacit, unexpressed or, for example, covert knowledge. We have only partial knowledge of how it is created (in human mind and in the lives of organisms). Some think that 80% of our knowledge belongs to this category. This knowledge is manifested in skills and its elements can only be mastered through practice, usually in informal circumstances. In many cases even those possessing it do not know about it. János Bolyai classifies motivation, too, as something that belongs to this type of knowledge.

In the bottom right vertex of the triangle above we can see the ability to want. We have limited knowledge of how free will influences the range of our emotions. We have not discovered as yet how community will can be shaped by an individual or the community so as to reach a happier future. The free will of the individuals is regulated by the groups they belong to, as well as society and its institutions and the accessible sources of information. János Bolyai thought the only feasible way was developing human knowledge; By developing the empirical knowledge of the individual he will be capable of the positive development of his free will, and, if community will is liberated from the compulsions restricting it, he will be capable of becoming a member of a grassroots community.

When we learn something, our knowledge creates a new level, in other words, we skip to the next level in a discrete way, until we reach, in the vertex of the tetrahedron, the One, the real, true knowledge. This is represented by the Figure below and in the quotation from János Bolyai's work: *„…véleményem, sőt okos hitem, sőt meggyőződésem szerint [valamint valahogy tudó nem érezhet életje mozgásában, s nem akarhat (a tudásra nézve) egy élőlény sem] … a tudás az érzés és akarás (egy mozgással csak) mind e három*



a test állapotja által is. S ez viszont (de ha a tudás mozgás) úgy per se, pertinens! De így még neve: sincs a szellemnek. Szabad azonban 'tudó, érző, akaró' szellemet érteni… s a test által (meg van határozva) tud, érez, és okos és mint vont kő esik a földre, vagy nyomja az alatti szert". [2: JB 53/4]

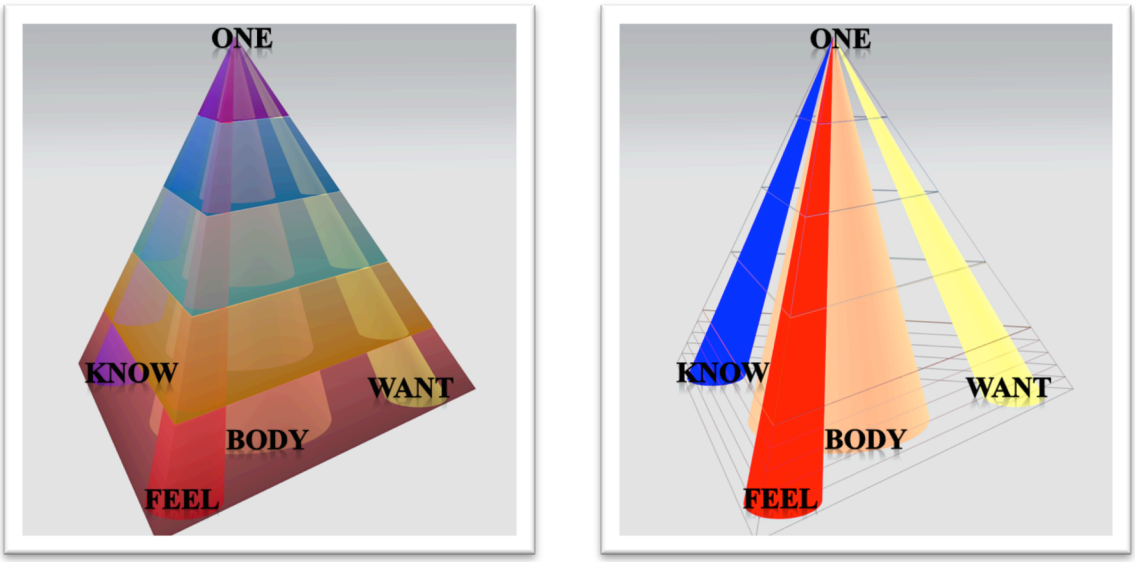


Figure 4: The tetrahedron and the 3+1 model with the One.

Hermes Trismegistos, of the traditional school of thought, interprets the One the other way round; *"And as all things have been & arose from one by the mediation of one: so all things have their birth from this one thing by adaptation."* (The 3rd statement of the Emerald tablet, as translated by Isaac Newton) [8].

During logical modelling on the three lateral faces of the tetrahedron we transform the mathematical image on the lateral faces ABC so that it is assembled in its vertex in a conceptual image. János Bolyai imagined the space of non-Euclidean geometry on the surface of a hemisphere with a finite radius, which does not include the great circle of the sphere. The tetrahedron shapes grow out of this hemisphere surface.

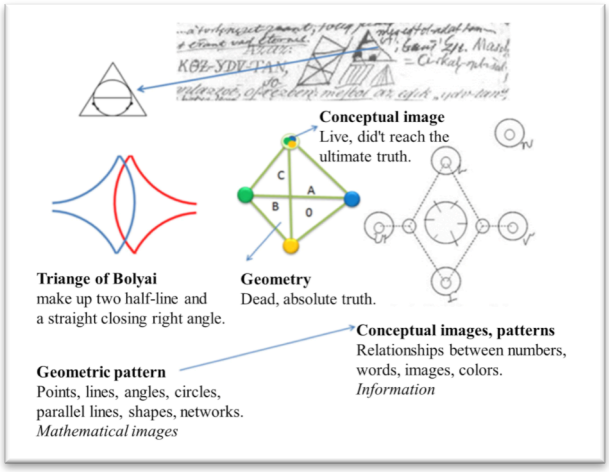


Figure 5. The tetrahedron as the shape of relations between conceptual images [9].

The application of the shape of the tetrahedron was inevitable in János Bolyai's methodology. This is proved by his drawing drafted on the first page of his *Köz-Üdv Tan* (Public welfare science) within his work titled *Tan* (Science) (see at the top of the previous Figure). In this drawing he represents that on the base of the tetrahedron it is possible to build models according to all the three types of geometry (Euclidean, non-Euclidean and absolute geometry).

The process of modelling, however, takes place in the human mind. Bolyai indicated this by the inner circle of the triangle; here is the body of the individuals. During learning the human, his body and memory undergo changes.

In the modelling process, one chooses patterns from his mind according to his knowledge, experience and will, using a "cruising opening". Searching while cruising he gets higher and higher on the A, B and C lateral faces of the tetrahedron.

With the help of the tetrahedron we can examine the learning/problem solving of one person, a community or mankind. There are smaller and larger tasks, such as learning a thesis, a lesson or completing a whole course or training.

According to János Bolyai, the role of the spirit is to preserve all the information, events, knowledge, experience, physical and mental state that happened to somebody during his life. Today we call this ability memory. As Bolyai put it, *„Eséréstől vagy esméret miért e ı=ı avval, s hasonlókép tudat ı=ı tudás-sal, érzés (s) akarás elválhatatlanak, mégpedig úgy van elegyitve vagy együtt vagy együtt egykor van e három, hogy csak egy (lélek); s a"lélek"nek neve szem csak a 3-at egy-itve, vagy a háromnak együlését, tudását [mit helytelenül belső érzéknek neveztek eig…] De érzés talán van esméret, s akarat nélkül, s talán minden anyag bír ollyal."* [2:JB 73-8]

Time is not interpreted in the approach of geometry but János Bolyai could not exclude it from the process of modelling because mathematical and logical modelling requires a view of time. Both in the whole process of modelling and in a given tetrahedron, for the production of mathematical and conceptual images, that is, for reaching the aim there is a time limit. The time needed depends on the nature of the problem. The progress of time defines the achievement of the goal of modelling, its success. Considering that moving from one tetrahedron to the other means a qualitative leap, it is also a leap in time. Jumping to another tetrahedron level or from one tetrahedron to the other is without a transition (can be described with a unit step function).

If we find a contradiction while using Bolyai's model, then we also have the solution in his geometrical model: two things or vertices can be dissolved in the third one. So that mankind will be able to solve its own problem, namely that the majority of people live in oppression, the contradiction that hinders harmony between human communities should be eliminated. There are three such obstacles: the lack of knowledge, of feeling and of will.

The first obstacle is the lack of knowledge, that is ignorance, which means that people solve everything based on their feelings and not on knowledge. As a consequence, the external social order exploits the majority of society (in Bolyai's time the serfs). This obstacle makes the people live for their range of emotions and not think about doing something against the external social order. They resign themselves to their state. The solution is giving them knowledge so that they can see that the ruling stratum/the one having political power deceives them because of their own living conditions, etc. So that the majority will be able to eliminate the contradiction between knowledge and their range of feelings (know-feel), they should have free will, independent of the demands of society.

The second obstacle is the lack of will, which means it is no use knowing something if you do not want to change the range of your feelings. In other words, knowledge cannot change someone's range of feelings. If someone grows into the range of feelings of their environment, they do not have the will to want to feel differently.

The third obstacle is the lack of feeling/practice. If you can create a balance between feeling/practice and will, you will be able to learn. You need free will so that you will be able to develop your knowledge. At present, it is free will that social demand wants to supress, an attack educational communities already stand up to.

János Bolyai designed an educational system for the majority of society that was built on the serfs' experience. At the same time, for educated people, he wanted to create a scholarly space that would involve all the bases of sciences: academies of sciences, libraries and all the religious places where teaching takes place. These places would have been connected by mental telegraphic lines (today it is the Internet). Teachers who would have

undertook teaching the people would have been able to use this knowledge network and Bolyai's work titled *Tan* (Science*)* as the source of knowledge.

János Bolyai writes that a teacher should have 12 learners. The teacher saves the learner from the problem of feeling; He assigns the learning material, the method and provides practical tasks, etc. needed for learning. In a society you must live up to the expectations of a community but the teacher switches off the demands of society. As Bolyai says it, he is a propagator.

Bolyai says it is possible to work only with young people because the elderly have already resigned themselves to their fate. He thinks school education should start in single-sex schools, at the age of 7. Boys need to study up to the age of 24, but girls may stop at the age of 16-17. Work for adults should be organised in such a way that they could spend at least 2 hours a day with learning.

He divided the Earth into 12 regions and thought that all books worth reading should be collected in an abridged form and made accessible to everyone. Those writing what others have already written down or some nonsense should be put to shame.

In the 21st century we have the conditions needed for the teaching/learning conducted according to the modelling based on János Bolyai's recommendations at our disposal. It is time to start, because, as Bolyai said, if it does not happen, *"...mankind may come off badly after 2000 years"* [15:132].

III. SOME CHARACTERISTICS OF TODAY'S EDUCATION, HAVING THE KNOWLEDGE OF THE BOLYAI MODELLING

A. Networking in our age and the Bolyai modelling

By the beginning of the 21st century technical and scientific knowledge that János Bolyai could not even imagine have been developed. Control exercised from above, network systems have been formed and by now the whole Earth has been covered with the information superhighway. While in the organisation of top-down networks all the arsenal of management methods can be implemented and the users are prepared by a multitude of schools for modelling conceptual images, in grassroots networks, apart from technical options, the individuals' self-governing knowledge and learning methods fall significantly behind technical options.

In theory everyone has at their disposal (scale-free[4]) communications networks needed for grassroots modelling systems that unite mankind and programmes for content sharing. The opportunity for the modellers is given to truly get to know their environment, to share new knowledge with others, to teach other modellers how to learn about truth. But individuals cannot or are only partially capable of using grassroots systems for improving their own living conditions, for small systems' self-organisation. They lack modelling knowledge.

The attitude of mankind towards knowledge has also been changed. Along with system approach, cybernetics, computer science and other sciences, the science of knowledge, in other words cognitive science has been developed, which we should rather consider a cognitive approach. The common method for all these sciences was provided by cybernetics and spread by computer science. From the middle of the 20th century, the 'input-output' method of cybernetics was widely applied in the area of artificial intelligence, robotics research, the application of computers as well as the modelling of the coordination of human thinking. The research in artificial intelligence developed modelling processes based on rules and explanations. All these methods were adopted and applied by the Internet on the one hand and by robotics research on the other.

The difference between the modelling method of cybernetics and that of Bolyai is that while cybernetics modelling is mainly built on the approach of the Euclidean homogenous, infinite space and transmits knowledge primarily for serving those managing societies and organisations, Bolyai's modelling is based on the view of non-Euclidean finite but boundless space. There is a condition for acquiring real knowledge, namely the acceptance of the fact that what is inside is like what is outside and what is outside is like what is inside.[5] In other words, the contradiction between the external and the internal must be balanced in the present again and again, which is the duty of educated people. This insight of Bolyai was justified by modern psychology and brain research in the statement that human mind communicates with its environment and with its own internal word using the same concepts. That is why it can produce the archetype of its actions, which serves for directing those actions. However, so that it will be real, this archetype must be built on self-recognition. In our schools, however, students are not prepared for that. János Bolyai created the 'paradigm' of human cognition, which can only be understood by the teachers of the 21st century, as – more and more of them – teach according to the connectivist approach.

B. The evaluation of the modern educational methodologies using Bolyai's model

"The underlying three great learning theories of educational methodologies are behaviourism, cognitivism and constructivism. All of them were developed before the technological revolution. In the last 20 years technology has rearranged our way of life, communication habits and the way we learn. The pedagogical guidelines defining our learning habits have always been shaped by the current socio-cultural impacts." [10] These three theories are present in the methodology of pedagogy and in educational systems but we do not know about the interaction between these theories, although the unity of

[4] Scale-free network: A complex network is a scale-free network if its nodes' degree distribution follows a power law. There is a decreasing number of an increasing degree of nodes, in compliance with power law. These networks have a relatively high number of high-degree nodes, and the distribution of node degrees does not depend on the size. Typically, as the number of connections increases one order of magnitude, the probability of high-degree node incidence decreases one order of magnitude. [16]

[5] This is an analogy to the 3rd statement of Hermes Trismegistos' Emerald Tablet, saying: "That which is below is like that which is above & that which is above is like that which is below to do the miracles of one only thing." (Translation by Isaac Newton [8]). Its interpretation is: Everything is created by the infinite, impersonalised creativeness and everything returns to it.

theory and practice of pedagogy is constructed in these interactions. Because of the trinity of Bolyai's model, it can be used to answer the many questions that arise. Let us sum up the learning theories in a nutshell and examine in which vertex of the triangle base of the tetrahedron we could place them.

Behaviourism; John B. Watson suggests [11] that we should focus on examining behaviour for the sake of objectivity. He thinks that human behaviour can only be objectively discerned and described by an observer. – In Bolyai's model this is 'want'.

Cognitivism; According to Karl Popper we do not have time for a debate over the definitions of concepts. It is more important to contemplate significant problems even at the cost of a word having different meaning contents depending on its use and user. – In Bolyai's model this is 'feel'.

Constructivism; Learning and knowledge is an active internal construction process, including its outcome. The recipient has the same key role in its formation as the one who conveys it. This observation has a number of profound theoretical and practical consequences. – In Bolyai's model this is 'know'.

The place and content of these three educational methodologies completely coincide with what Bolyai established about the structure of knowledge in his work *Tan* (Science*)*. Consequently, in Bolyai's chain of thought a hundred years' development of pedagogical theory can be modelled and evaluated and the objectives and tasks following from this model can be defined. In the Figure below the question marks signify the possible areas of development.

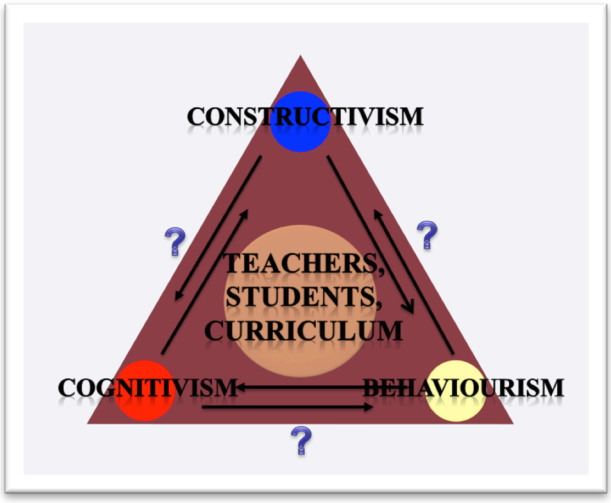


Figure 6: Educational methodologies of our age and the possible areas of development in Bolyai's model.

Now we can start developing the methodology of pedagogical modelling, including the framework of the methodology of modelling. The first axiom is that the three methodologies follow from one another and they are of equal importance. Each of them has its role in the education and teaching of the members of society and they can only be applied together. We must not make the mistake committed by leading mathematicians at the beginning of the 20th century, who interpreted János Bolyai's geometric system as three independent systems (Euclidean, hyperbolic and absolute geometry) and examined them separately.

Our pedagogical system of approach should be directed to change and develop the child's/student's/adult's abilities and not at the learning material. Teachers supporting reforms in pedagogy have always wanted to apply new methodologies and those opposing them have always wanted to insist on the old ones. There has never been a comprehensive investigation to model the clear objectives of education and training and the process of transforming a child into an adult. What we have is a number of unfinished ideas that showcase opportunities provided by technology for their own sake.

Applying network theory, connectivism combines the other three methodologies and organises them into a unity. This is what Kerr (2007) thinks, too, saying that, although technology affects learning environments, learning theory has not created anything new as compared to existing theories. Also, certain scientists say that connectivism is not a learning theory but a pedagogical approach to training [12].

C. Teaching according to the Bolyai-model

Bolyai says that while teaching, one has to reach the vertex of several tetrahedrons (see the following Figure) – at least of three. In further phases of learning other tetrahedrons are joined to them as needed.

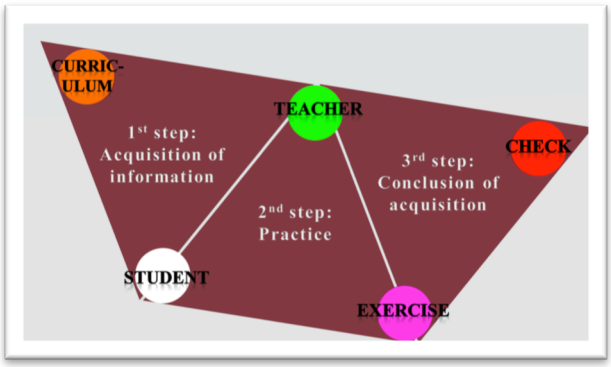


Figure 7: The process of teaching-learning with Bolyai's triangles.

The base triangle of the 1st tetrahedron/1st step is the transfer of information. The teacher is positioned in the top vertices of the triangles. His job in this step is to transfer/convey/make the learners acquire information, his role is directing the learners. There must be learning material ready to be taught (in the top left vertex). The job of the learner(s) (in the bottom vertex) is to consciously acquire the material.

The base triangle of the 2nd tetrahedron/2nd step is practice. The teacher facilitates the practical application of what has been learnt. The learner practises what has been learnt by solving problems. (These two vertices coincide with those of the previous triangle.) The third vertex represents the exercises.

The base triangle of the 3rd tetrahedron/3rd step is the conclusion of acquisition. This is how a lesson, a course or a training is completed. Its two vertices are those of the teacher and the exercise of the 2nd base triangle and the third is testing. Testing also involves reinforcement of what has been learnt and the teacher, learner, community get feedback about the success of the learning process. Preferably, the teacher and the learner have reached their

objectives and learning is concluded. Otherwise they need to move along the appropriate levels on the faces of the 1st, 2nd and 3rd tetrahedrons again.

At the bottom of the triangles there is the 'body', which is both the information and its carrier. The change brought about by learning alters/modifies the person. In each of the triangles the learner completes a level and gets higher and higher in the tetrahedrons, into the vertex of knowledge, the only one thing, true knowledge.

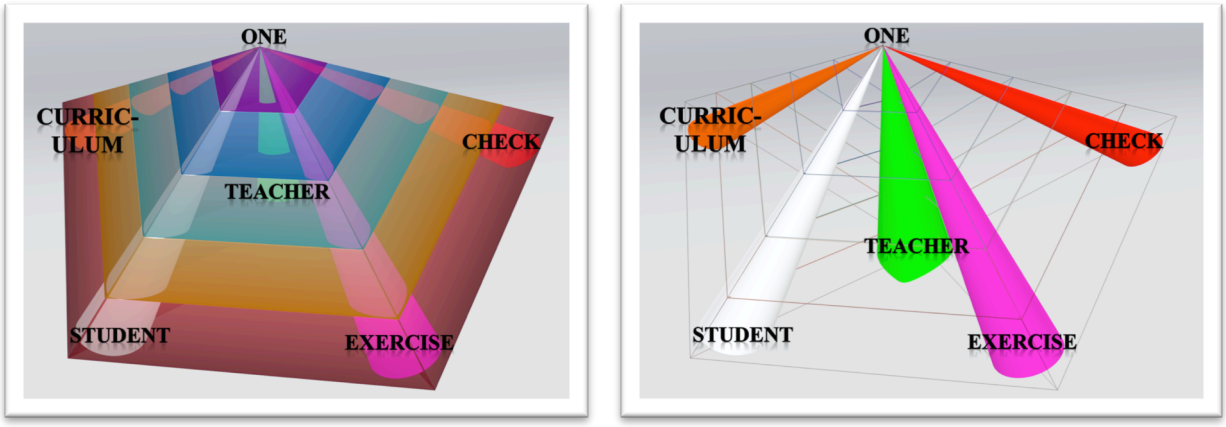


Figure 8: The process of teaching-learning with tetrahedrons illustrating the levels of knowledge.

The teacher should, bearing in mind the learner's previous experience and learning style, prepare the lesson, considering whether the learner has implicit and/or explicit (practical or theoretical) knowledge and learning style. The teacher should monitor the learner's progress and correct it if he sees that the vertices or the faces of the triangle are not identical. Knowledge taking shape inside the learner should be applicable in practice. Unfortunately, at present we mainly teach knowledge, teachers do not use the implicit method, they do not prepare learning material in this way.

IV. A REVISED BOLYAI TEACHING MODEL AND ITS APPLICATION

A. A revised Bolyai teaching model

As mentioned earlier, there is a condition for acquiring real knowledge; the acceptance of the fact that what is inside is like what is outside and what is outside is like what is inside. In other words, the contradiction between the external and the internal must be balanced in the present again and again, which is the duty of educated people. By including this, we also consider the external environment of the individuals. We are going to add three more subtopics to the ones already selected, such as society, family and knowledge (see Figure 10). Now the connections and contradictions between the new pairs of subtopics can also be examined and resolved in the third subtopic.

These three main constituents of the environment are the key inputs of the teaching-learning process, too, which is examined based on cybernetics, outlined in Figure 9. These constituents could also be examined as systems but now we will consider them as black boxes. "The environment is too complex, therefore I consider that many other areas are also included in these three." [13]

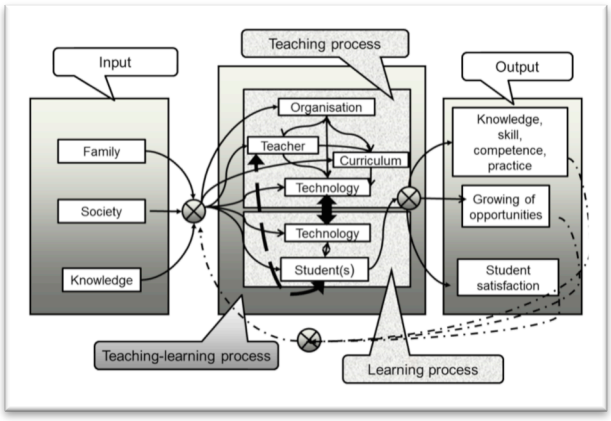


Figure 9: The structural model of the teaching-learning process [13].

The internal order is the balance of '*know-feel-want*', carried by the minds of the individuals. This balance should correspond to their own bodily capabilities. This is where one solves conflicts and forms a unity to eliminate the threefold contradiction of the personal '*know-feel-want*' (see next Figure). The internal and the external order should agree, in order that one can function efficiently.

Scientific literature differentiates practical, limbic and reptilian brain [14]. 'Know' (implicit cognition of the environment) is linked to the practical brain. 'Feel' (organising elements into a system) is connected to the practical brain and 'want' (technology, tools, resources) to the reptilian brain. This classification does not mean that the limbic brain does not have a will. The limbic brain overrules the will of the reptile brain. We make them work in the experiment described below.

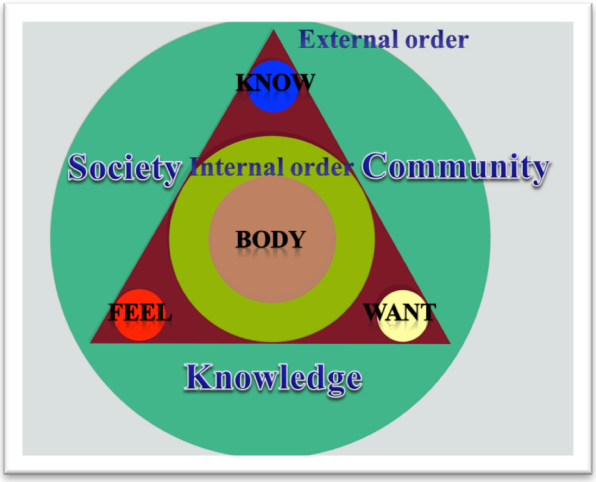


Figure 10: The Bolyai-model augmented with internal and external order

One tries to adjust his internal order to the external order, too, but it also influences the order of the environment. New information, experience, etc. needs to be inserted in our internal order. Today we call this mind mapping.

The external order is the unity of the (school) requirement of the social-community-knowledge trinity. They must conform to one's internal order, must correspond to it. Also, the topic/task examined by the person must live up to the expectations of society. The execution of the topic should be suitable for the family (friends', working, village, city, country, etc,) community. Applied knowledge should be appropriate for academic knowledge, whose carriers are the school, the teachers' staff, the individual teachers, the master and the library.

B. An experiment to prove the Bolyai modelling

János Bolyai debveloped this modelling by experimenting on himself. Its understanding, too, depends on self-knowledge; the model should be examined from within. We suggest that everyone carries it out. Its main steps are as follows.

1. Take a few sheets of blank paper or open the word processor (and save the document). Make a table with two columns and write the names of the vertices of the triangle in the heading. If done correctly, 'know' will be placed in the first column and 'feel-want' in the second.
2. Specify a current task (it should be a real problem, relevant for you). It could be related to the community, school, work or our private life.
3. We surely have the knowledge to interpret the task and we also want to complete it.
4. What do we need to interpret the task? Experience, that is the patterns of the feeling we had when carrying out similar tasks, stored and labelled, indicating how we felt then.
5. What do these 'feel patterns' cause? We either have an uncomfortable feeling about the task or we are happy about it. Whether we interpret the task even despite an awkward feeling depends on whether we really want to do it or not. Whether we want to, in the case of an externally prescribed task, also depends on coercive rules attached to the task. (Let us now suppose that we cannot refuse to do it.)
6. Sooner or later, there will be no obstacles to interpret the task. The features of *know-want-feel* will be in agreement.
7. We need to examine whether there are empirical patterns and sufficient knowledge to carry out the task. Therefore we need to enumerate the patterns and put together the idea which includes
   - the task reformulated into an objective,
   - the resources needed for the task,
   - the actions to be carried out,
   - what we have at our disposal and what needs to be obtained from the environment.

   In this phase, the three abilities (*know-feel-want*) are constantly making compromises in our mind. Two are in agreement but the third one will regularly block the process, which can be represented in a tetrahedron. The plan of how to carry out the task is in its vertex.
8. At this point, the idea has not been analysed, it has not been tested if it will really lead to the planning of the action needed to carry out the task. We need to go through what we have written. Firstly, we need to check if free will hinders the execution of the task or if the bodily balance collapses. Secondly, we need to check if there is a contradiction between experience and knowledge. If there is, can it be eliminated before the execution, and what needs to be done to achieve it? Finally, what impact will the execution of the task have on the community.

Without elaborating on it any further, after building the simplest regular body to facilitate teaching-learning, then the building of the regular body to plan the execution of the task and then to control the execution would follow.

To prove the internal order in practice, the first author of this paper documented the solution of a work problem in a table, demonstrating how she reached the top vertex of the tetrahedron. She thought she would describe in sensible steps in the left-hand column what to do. In the right-hand column she would write the related 'know' (explicit knowledge needed for the solution), 'feel' (practical knowledge, feelings), 'want' (the specified objective) and the 'body' (how her body reacts while carrying out the task). During the process, she was aware of the external environment and was documenting it.

She did not manage to carry out the task as planned though, because in the left-hand column there were not only the things to do but also the documentation of the process Bolyai called the cruising along the faces of the tetrahedron. As time went on, she became more and more engrossed in the process and, being in a kind of flow state, she did not keep the rule of putting only the four aspects (*know-feel-want* + body) formulated as facts. In fact, more and more tasks to be done were written down.

The experiment – apart from the lessons drawn – finished in a cheerful mood; the fabric of associations, the flitting ideas in the columns were clear, the style of the composition was spot-on. To put it simply, a funny story was created. The content of the bottom right cell was the following:

„1. I know I planned to finish the work here. I'm ready.

2. It was amusing to read how my opinion changed about how I work. Now I know that there's an organised hubbub, a colourful and noisy carnival in my head.

3. I want to send this product to uncle Imre quickly. I'll call him right now to warn him that I've sent him a message. While he's reading, I'll be writing letters to my students.

4. Or I'll start preparing dinner ☺.

1. It's obvious that the contents in the two columns've got mixed up; I've got activities here and the *know-feel-want* + body ideas on the left-hand side." (It is not by accident that this point was also numbered as 1; this is what the author realised after reading the documentation.)

V. SUMMARY

Examining specialised pedagogical literature we learn that the best scientists are able to elaborate on their ideas in the dynamism of the contradictory or contradiction-free relations of the three elements. People want to fight the past, trying to prove the theories of the new. In reality, the past is within us and if we do not learn how to use it, it will result in the end of mankind. This is what János Bolyai wanted to teach to mankind. The problem lies in the capability of human memory. Without doing exercises with their working memory, students and teachers are not capable of modelling on the 3+1 (*know-feel-want* + body) model. We think that the Bolyai-modelling could efficiently support the development of schools, the preparation for new problems and the creation of new school and teachers' communities. Those learning in an environment applying this model live up to modern leadership requirements.

In this paper we gave an overview of Bolyai's model after presenting his view of the world. After that, we modelled the educational methodology of our age in the tetrahedron model of Bolyai to point out how the learning theory underlying the three most significant methodologies of last century, applying Bolyai's methodology, is combined by the network theory, connectivism. We also presented our upgraded Bolyai model, which can be used as a starting point for teaching. Finally, to prove the concept of internal order, we presented Bolyai's method, which he developed by experimenting on himself. We also conducted a practical experiment, the conclusions of which we summarised.

The new features of the paper were also illustrated by figures, enumerated here as a final summary:

- The Euclidean plane, the hyperbolic plane and the absolute geometric plane constructed from them (Figure 1);
- The triangle base of Bolyai's tetrahedron with empirical knowledge, free will and group will indicated (Figure 3);
- The tetrahedron and the 3+1 model with the One (Figure 4);
- Educational methodologies of our age and the possible areas of development in Bolyai's model (Figure 6);
- The process of teaching-learning with Bolyai's triangles (Figure 7);
- The process of teaching-learning with tetrahedrons illustrating the levels of knowledge (Figure 8);
- The Bolyai-model augmented with internal and external order (Figure 10).

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